

reaction it is sufficient to measure only the components  $\rho'_{-1}$ ,  $\rho'_0$ , and  $\rho'_{+1}$  of the statistical vector of particles b. Equation (12) then becomes

$$\sigma'_D(\vartheta, \varphi) - \sigma_D(\vartheta, \varphi) = F^2 p_c^2 p_a^{-2} (\rho'_{+1} \rho_{-1} - \rho'_0 \rho_0 + \rho'_{-1} \rho_{+1}) = -F^2 p_c^2 p_a^{-2} (\rho'(\vartheta, \varphi + \pi) \rho). \quad (13)$$

Since the beam and target are unpolarized in (10), it can be shown that the statistical vectors of particles b emitted at angles  $\vartheta, \varphi$  and  $\vartheta, \varphi + \pi$  are of equal magnitude and oppositely directed (and both perpendicular to the reaction plane). In place of  $\rho'$  let us introduce the polarization vector  $\mathbf{N}'$  (which for particles of spin  $1/2$  is equal to twice the expectation value of the vector spin operator), so that  $\rho'(\vartheta, \varphi) = \mathbf{N}'(\vartheta, \varphi) \sigma_I(\vartheta, \varphi)$  (where  $\sigma_I(\vartheta, \varphi)$  is the cross section for the inverse reaction with unpolarized c and d). Rewriting (13) (we note that  $\rho = \mathbf{N} \cdot \mathbf{1}$ ) in the form

$$\sigma'_D(\vartheta, \varphi) - \sigma_D(\vartheta, \varphi) = F^2 p_c^2 p_a^{-2} (\mathbf{N}'(\vartheta, \varphi) \rho) \sigma_I(\vartheta, \varphi), \quad (14)$$

we obtain a relation for arbitrary spins which has often been obtained in the literature for various special cases (see, for instance, Lapidus<sup>7</sup> and the literature he refers to).

The author expresses his gratitude to L. I. Lapidus for discussing the present work.

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## ON THE METHOD OF "SECOND QUANTIZATION" IN PHASE SPACE

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In order to give a statistical description of processes in systems of interacting particles we use as independent variables the numbers of particles in different points of coordinate-momentum phase space, which at every point of phase space are random functions of the time. We give classical and quantum mechanical equations for these functions; from these equations we can, by averaging the random functions over their distributions, obtain a set of equations for the classical and quantum mechanical distribution functions. To illustrate the possible use of this method we obtain expressions for the excitation spectrum and correlation function for systems of particles interacting through central forces in the case where close interactions are unimportant. This method is convenient for considering system of particles and fields, in particular, electromagnetic interactions. It is possible to generalize this method for the relativistic case.

**I**N the present paper we give a method to introduce collective variables to describe a system of interacting particles which is slightly different from the considerations in the papers of Tomonaga,<sup>1</sup> Bogoliubov and Zubarev<sup>2,3</sup>, and Bohm and Pines.<sup>4</sup> We introduce as independent variables the numbers of particles in

different points of coordinate-momentum phase space. In this sense this method is analogous to the method of second quantization and we have therefore called it in Ref. 5 the method of second quantization in phase space.\* The method under consideration is convenient because it makes it possible in a number of cases to describe in a unified and fairly simple way processes which occur both in classical and quantum mechanical systems. We give in the present paper, on the basis of this representation, conclusions derived from a set of equations for classical<sup>6</sup> and quantum mechanical distribution functions.<sup>7</sup> It is shown how this method can be used to investigate the excitation spectrum and correlation function of systems of interacting particles.

We consider a classical system of  $N$  particles interacting through central forces. We denote by

$$N_{\mathbf{q}\mathbf{p}}(t)d\mathbf{q}d\mathbf{p} = \sum_{1 \leq i \leq N} \delta(\mathbf{q} - \mathbf{q}_i) \delta(\mathbf{p} - \mathbf{p}_i) d\mathbf{q}d\mathbf{p}$$

the number of particles in a volume element of phase space  $d\mathbf{q}d\mathbf{p}$ . The Hamiltonian function of the system under consideration can in the classical case be written in the following form,

$$H = \sum_{1 \leq i \leq N} \frac{p_i^2}{2m} + \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \Phi(|\mathbf{q}_i - \mathbf{q}_j|). \quad (1)$$

Using the definition of the quantities  $N_{\mathbf{q}\mathbf{p}}(t)$  we can write expression (1) in the following different, equivalent form,

$$H = \int \frac{p^2}{2m} N_{\mathbf{q}\mathbf{p}} d\mathbf{q}d\mathbf{p} + \frac{1}{2} \int \Phi(|\mathbf{q} - \mathbf{q}'|) \{N_{\mathbf{q}\mathbf{p}} N_{\mathbf{q}'\mathbf{p}'} - \delta(\mathbf{q} - \mathbf{q}') \delta(\mathbf{p} - \mathbf{p}') N_{\mathbf{q}\mathbf{p}}\} d\mathbf{q}d\mathbf{q}'d\mathbf{p}d\mathbf{p}'. \quad (2)$$

It is convenient to choose as the initial Hamiltonian function a function which includes the terms  $\Phi(\mathbf{q}_i - \mathbf{q}_j)$  for  $i = j$ . In that case we obtain instead of expression (2) the following expression for the Hamiltonian function

$$H = \int \frac{p^2}{2m} N_{\mathbf{q}\mathbf{p}} d\mathbf{q}d\mathbf{p} + \frac{1}{2} \int \Phi(|\mathbf{q} - \mathbf{q}'|) N_{\mathbf{q}\mathbf{p}} N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}d\mathbf{q}'d\mathbf{p}d\mathbf{p}'. \quad (3)$$

writing the Hamiltonian function in the form (3) presupposes that all particles are identical.

The functions  $N_{\mathbf{q}\mathbf{p}}(t)$  are for all fixed values of  $\mathbf{q}$  and  $\mathbf{p}$  random functions of the time. We denote by  $F(\dots N_{\mathbf{q}\mathbf{p}} \dots)$  the multidimensional distribution law for the different values of the  $N_{\mathbf{q}\mathbf{p}}(t)$ . In the following we shall define averages using the function  $F(\dots N_{\mathbf{q}\mathbf{p}} \dots)$ .

Using expression (3) for the Hamiltonian function we find the equations of motion in the representation under consideration,

$$\dot{\mathbf{q}} = \frac{\partial}{\partial \mathbf{p}} \frac{\delta H}{\delta N_{\mathbf{q}\mathbf{p}}} = \frac{\mathbf{p}}{m}, \quad \dot{\mathbf{p}} = -\frac{\partial}{\partial \mathbf{q}} \frac{\delta H}{\delta N_{\mathbf{q}\mathbf{p}}} = -\frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q} - \mathbf{q}'|) N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}'d\mathbf{p}'. \quad (4)$$

Equations (4) combined with the equation of continuity in phase space  $dN_{\mathbf{q}\mathbf{p}}(t)/dt = 0$  leads to the following equation for  $N_{\mathbf{q}\mathbf{p}}(t)$ ,

$$\frac{\partial N_{\mathbf{q}\mathbf{p}}}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial N_{\mathbf{q}\mathbf{p}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q} - \mathbf{q}'|) N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}'d\mathbf{p}' \frac{\partial N_{\mathbf{q}\mathbf{p}}}{\partial \mathbf{p}} = 0. \quad (5)$$

Equation (5) has the same form as the classical self-consistent field equation which was investigated in detail by Vlasov<sup>9</sup> and Landau.<sup>10</sup> However, there is an essential difference between the two equations. Indeed, the classical self-consistent field equation is an approximate equation for distribution functions which can be obtained from equations for the distribution function of the whole system  $f_N(\mathbf{q}_1, \dots, \mathbf{p}_1, \dots, t)$  assuming this function to be factorized

$$f_N = \prod_{1 \leq i \leq N} f_1(\mathbf{q}_i, \mathbf{p}_i, t),$$

or as a first approximation in an expansion in powers of a small parameter.<sup>11</sup> The self-consistent field equation does not take into account the correlation produced by the interaction between the particles.

\*M. Schönberg<sup>8</sup> has published under an analogous title several interesting papers, but his aim and results are different and we shall therefore not discuss the results of these papers.

Equation (5), however, is an equation for the random functions  $N_{\mathbf{qp}}(t)$ . As will be seen presently we can, by averaging by means of the phase-space distribution function  $F(\dots N_{\mathbf{qp}} \dots)$ , obtain from Eq. (5) a set of equations for the distribution functions  $f_1(\mathbf{q}, \mathbf{p}, t)$ ,  $f_2(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t)$ , which are considered in the well-known monograph of Bogoliubov's<sup>6</sup> and we can therefore consider correlations in every approximation. The relation between Eq. (5) and the self-consistent field equations for the distribution functions is analogous to the relation between the equations for the state function  $\Psi$  in the theory of second quantization and the Hartree equations. We must finally note that in Vlasov's papers<sup>9</sup> also another interpretation of the equation for the function  $f_1(\mathbf{q}, \mathbf{p}, t)$  was suggested.

We shall now establish the connection between the average of the product of functions  $N_{\mathbf{qp}}(t)$  in different points of phase space and the functions  $f_1, f_2$ . By comparison with the expressions for the average values of functions of dynamical variables which are additive, binary, etc. averages, obtained by using the distribution functions  $f_1, f_2$ , and the function  $F$ , one can obtain the relation between the distribution functions and the average values of the products of the  $N_{\mathbf{qp}}(t)$ . In the classical case these relations have the following form,

$$Nf_1(\mathbf{q}, \mathbf{p}, t) = \overline{N_{\mathbf{qp}}}(t); \quad (6)$$

$$N(N-1)f_2(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}', t) = \overline{N_{\mathbf{qp}}N_{\mathbf{q'p'}}} - \delta(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}')\overline{N_{\mathbf{qp}}}; \quad (7)$$

$$\begin{aligned} & N(N-1)(N-2)f_3(\mathbf{q}, \mathbf{q}', \mathbf{q}'', \mathbf{p}, \mathbf{p}', \mathbf{p}'', t) \\ &= \overline{N_{\mathbf{qp}}(t)(N_{\mathbf{q'p'}}(t) - \delta(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}'))(N_{\mathbf{q''p''}}(t) - \delta(\mathbf{q}'-\mathbf{q}'')\delta(\mathbf{p}'-\mathbf{p}'') - \delta(\mathbf{q}-\mathbf{q}'')\delta(\mathbf{p}-\mathbf{p}''))}. \end{aligned} \quad (8)$$

From Eqs. (6) to (8) one can easily find the inverse expressions of the average values of the products of the  $N_{\mathbf{qp}}(t)$  in terms of the functions  $f_1, f_2$ . The functions  $f_i(\mathbf{q}_1 \dots, \mathbf{p}_1 \dots)$  are normalized as follows,

$$\int f_i(d\mathbf{q}_i)(d\mathbf{p}_i) = 1.$$

In Eqs. (6) to (8) one can take the limit  $N \rightarrow \infty$ . In that case it is convenient to use instead of the functions  $N_{\mathbf{qp}}$  quantities referring to unit volume or to one particle. If we consider the quantities  $N_{\mathbf{qp}}(t)$  at different points in time, we obtain the many-time distribution functions which were considered by van Hove<sup>12</sup> and Tolmachev.<sup>13</sup>

Using Eq. (5) and expressions (6) to (8) we can obtain a set of equations for the usual distribution functions  $f_1, f_2, \dots$  and for the many-time distribution functions  $f_1(\mathbf{q}, \mathbf{p}, t)$ ,  $f_2(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t, t')$ ,  $\dots$ . Indeed, averaging (5) and using (6) and (7) we get the first equation of the set which connects the functions  $f_1$  and  $f_2$ ,

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial f_1}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q}-\mathbf{q}'|) \left\{ (N-1) \frac{\partial}{\partial \mathbf{p}} f_2(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}', t) + \delta(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}') \frac{\partial}{\partial \mathbf{p}} f_1 \right\} d\mathbf{q}' d\mathbf{p}'. \quad (9)$$

The occurrence of an extra term is connected with our choice of Eq. (3) for the Hamiltonian function. To get the next equation of the set one must multiply Eq. (5) by  $N_{\mathbf{q'p'}}(t')$  and average, and so on.

To obtain the solution of several problems it is convenient to use not Eq. (5), but the equation for the difference between the functions  $N_{\mathbf{qp}}$  and their average values, that is, for the functions  $\delta N_{\mathbf{qp}}(t) = N_{\mathbf{qp}} - \overline{N_{\mathbf{qp}}}(t)$ . The mean square deviation is connected with the correlation function,

$$\overline{\delta N_{\mathbf{qp}}(t) \delta N_{\mathbf{q'p'}}(t)} = g(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}', t) + \delta(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}')\overline{N_{\mathbf{q}}}(t). \quad (10)$$

From Eq. (5) we get the following equation for the  $\delta N_{\mathbf{qp}}$ ,

$$\frac{\partial}{\partial t} \delta N_{\mathbf{qp}} + \frac{\mathbf{p}}{m} \frac{\partial}{\partial \mathbf{q}} \delta N_{\mathbf{qp}} - \frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q}-\mathbf{q}'|) \frac{\partial}{\partial \mathbf{p}} \{ \overline{N_{\mathbf{q'p'}}} \delta N_{\mathbf{qp}} + \overline{N_{\mathbf{qp}}} \delta N_{\mathbf{q'p'}} + \delta N_{\mathbf{qp}} \delta N_{\mathbf{q'p'}} - \overline{\delta N_{\mathbf{qp}} \delta N_{\mathbf{q'p'}}} \} d\mathbf{q}' d\mathbf{p}' = 0, \quad (11)$$

from which we can easily obtain an equation for the mean square deviation,

$$\begin{aligned} & \frac{\partial}{\partial t} \overline{\delta N_{\mathbf{qp}} \delta N_{\mathbf{q'p'}}} + \frac{\mathbf{p}}{m} \frac{\partial}{\partial \mathbf{q}} \overline{\delta N_{\mathbf{qp}} \delta N_{\mathbf{q'p'}}} - \frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q}-\mathbf{q}'|) \\ & \times \frac{\partial}{\partial \mathbf{p}} \{ \overline{N_{\mathbf{qp}}} \overline{\delta N_{\mathbf{q'p'}}} \delta N_{\mathbf{q''p''}} + \overline{N_{\mathbf{q''p''}}} \overline{\delta N_{\mathbf{qp}}} \delta N_{\mathbf{q'p'}} + \overline{\delta N_{\mathbf{qp}}} \overline{\delta N_{\mathbf{q'p'}}} \delta N_{\mathbf{q''p''}} \} d\mathbf{q}'' d\mathbf{p}'' = 0. \end{aligned} \quad (12)$$

These equations simplify considerably, if we neglect in (12) triple deviations, which is possible when the deviations are small, and in (11) the term  $\delta N_{\mathbf{q}\mathbf{p}}\delta N_{\mathbf{q}'\mathbf{p}'} - \overline{\delta N_{\mathbf{q}\mathbf{p}}\delta N_{\mathbf{q}'\mathbf{p}'}}$ . The neglecting of this term in (11) is possible when the interactions are weak, and when we are interested in phenomena for which close interactions play a minor role. The solutions of (11) and (12) given below are only correct under these assumptions.

We consider one particular, very important case when the average distribution of particles in phase space does not depend on time and is uniform in coordinate space, that is,  $\overline{N_{\mathbf{q}\mathbf{p}}}(t) = \overline{N_{\mathbf{p}}}$ . This case corresponds to statistical equilibrium and represents a random, uniform, stationary process.<sup>14</sup> The mean square deviation depends in this case only on the time difference  $|t - t'| = \tau$  and on the relative position  $\mathbf{q} - \mathbf{q}'$  of the particles, i.e.,

$$\overline{\delta N_{\mathbf{q}\mathbf{p}}(t) \delta N_{\mathbf{q}'\mathbf{p}'}(t')} \rightarrow M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}'). \quad (13)$$

The function  $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$  is connected with the space-time correlation function of the stationary, uniform process under consideration. With these assumptions Eq. (11) has the following form,

$$\frac{\partial \delta N_{\mathbf{q}\mathbf{p}}}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial \delta N_{\mathbf{q}\mathbf{p}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q} - \mathbf{q}'|) \delta N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}' d\mathbf{p}' \frac{\partial \overline{N_{\mathbf{p}}}}{\partial \mathbf{p}} = 0, \quad (14)$$

and the equation for  $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$  has the form,

$$\frac{\partial M}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial M}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} \int \Phi(|\mathbf{q} - \mathbf{q}'|) M(\tau, \mathbf{q}', \mathbf{p}', \mathbf{p}'') d\mathbf{q}' d\mathbf{p}'' \frac{\partial \overline{N_{\mathbf{p}}}}{\partial \mathbf{p}} = 0. \quad (15)$$

Let us consider the solution of the equation for  $M$  in the case where  $\tau = 0$ . If we take into account the connection between the function  $M(\mathbf{q})$  and the correlation function  $g(|\mathbf{q}|)$ , which we have normalized according to Ref. 6,

$$g(\mathbf{q}, \mathbf{p}) = M(\mathbf{q}, \mathbf{p}) \Big|_{\overline{n}} - \delta(\mathbf{q}) \overline{N_{\mathbf{p}}} \Big|_{\overline{n}}; \quad \overline{n} = \int \overline{N_{\mathbf{p}}} d\mathbf{p},$$

we get the following equation for the correlation function

$$\sum_{i=1, 2} \left\{ \frac{p_i}{m} \frac{\partial g}{\partial q_i} - \frac{\partial}{\partial q_i} \int \Phi(|\mathbf{q} - \mathbf{q}'|) g(\mathbf{q}', \mathbf{p}') d\mathbf{q}' d\mathbf{p}' \frac{\partial \overline{N_{\mathbf{p}}}}{\partial p_i} \right\} = \sum_{i=1, 2} \frac{\partial \Phi(|\mathbf{q}|)}{\partial q} \frac{\partial \overline{N_{\mathbf{p}}}}{\partial p} \frac{1}{n}. \quad (16)$$

Solving Eq. (16) we get

$$g(|\mathbf{q}|) = \frac{1}{(2\pi)^3 \overline{n}} \int \frac{v(k) F(k)}{1 - v(k) F(k)} e^{-i\mathbf{k}\mathbf{q}} d\mathbf{k}, \quad (17)$$

$$M(|\mathbf{q}|) = -\frac{\overline{n}}{(2\pi)^3} \int \frac{1}{1 - v(k) F(k)} e^{-i\mathbf{k}\mathbf{q}} d\mathbf{k}, \quad (18)$$

where

$$F(\mathbf{k}) = \int \mathbf{k} \frac{\partial \overline{N_{\mathbf{p}}}}{\partial \mathbf{p}} \frac{m}{k\mathbf{p}} d\mathbf{p}, \quad v(k) = \int \Phi(|\mathbf{q}|) e^{i\mathbf{k}\mathbf{q}} d\mathbf{q}. \quad (19)$$

Substituting the Maxwell distribution into (18) we get a result which agrees with the correlation function obtained in Ref. 3. If the interaction between the particles is a Coulomb interaction we get from Eq. (18) the Debye expression

$$g(q) = -(1/4\pi \overline{n} r_d^2 q) \exp(-q/r_d).$$

Let us now find an expression for the average energy in the case under consideration. Substituting  $N_{\mathbf{q}\mathbf{p}} = \overline{N_{\mathbf{p}}} + \delta N_{\mathbf{q}\mathbf{p}}$  into Eq. (2) for the Hamiltonian function and averaging we get the expression

$$\overline{H} = \frac{3N\kappa T}{2} + \frac{N^2 v(0)}{2V} + \frac{V}{2} \int \Phi(|\mathbf{q}|) M(|\mathbf{q}|) d\mathbf{q}.$$

Let us now consider the solution of the equations for  $\delta N_{\mathbf{q}\mathbf{p}}(t)$  and  $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$  for the case when  $\tau \neq 0$ .

The solution of Eq. (14) can be written in the form of an infinite sum of harmonic terms with random amplitudes and phases,

$$\int \delta N_{qp}(t) dp = (2\pi)^{-4} \int n_{\mathbf{k}, \omega} e^{i\omega t - i\mathbf{k}\mathbf{q}} d\omega d\mathbf{k}. \quad (20)$$

Substituting (20) into (14) we find from the condition that the latter has a solution that

$$1 = v(\mathbf{k}) \int \frac{k \partial N_p / \partial p}{(k p / m) - \omega} dp, \quad (21)$$

which determines for our case the dependence of the possible values of  $\omega_{\mathbf{k}}$  on the wave vector  $\mathbf{k}$ . Expansion (21) agrees with the expression for the distribution function obtained by Vlasov<sup>9</sup> and Landau<sup>10</sup> for the solution of the linearized self-consistent field equation.

Formula (21) determines the form of the space-time correlation function. This follows immediately from the solution of Eq. (15), if we look for this solution in the form

$$\int M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}') dp dp' = (2\pi)^{-4} \int f(\omega, \mathbf{k}) e^{i\omega\tau - i\mathbf{k}\mathbf{q}} d\omega d\mathbf{k}. \quad (22)$$

Substituting expansion (22) into (15) we find that we can find a solution only if (21) is satisfied. Without solving (15) we can find an expression for the function  $M(\tau, \mathbf{q})$  for a system of particles with Coulomb interactions by using (21) directly. From expression (21) we can for a given value of  $\mathbf{k}$  determine the frequency  $\omega_{\mathbf{k}}$  and the logarithmic decrement  $\sigma_{\mathbf{k}}$  of the vibrations corresponding to the Fourier coefficients  $n_{\mathbf{k}}(t)$  obtained from the expansion

$$\int \delta N_{qp}(t) dp = (2\pi)^{-3} \int n_{\mathbf{k}}(t) e^{-i\mathbf{k}\mathbf{q}} d\mathbf{k}. \quad (23)$$

In the case of long wavelengths we can immediately write down the expressions for  $\omega_{\mathbf{k}}$  and  $\sigma_{\mathbf{k}}$ , if we use Landau's result<sup>10</sup>

$$\omega_{\mathbf{k}}^2 = \omega_L^2 + 3r_d^2 k^2; \quad \sigma_{\mathbf{k}} = \omega_L \sqrt{\pi/8} (kr_d)^{-3} \exp(-1/2k^2 r_d^2). \quad (24)$$

Since  $\sigma_{\mathbf{k}} \ll \omega_{\mathbf{k}}$  the Fourier coefficients  $n_{\mathbf{k}}(t)$  satisfy the differential equation

$$\ddot{n}_{\mathbf{k}} + 2\sigma_{\mathbf{k}} \dot{n}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 n_{\mathbf{k}} = 0. \quad (25)$$

From Eq. (25) we find an expression for the spectral density of the correlation function  $M(\tau, \mathbf{q})$

$$f(\omega, \mathbf{k}) = A / [(\omega - \omega_{\mathbf{k}})^2 + 4\omega_{\mathbf{k}}^2 \sigma_{\mathbf{k}}^2], \quad (26)$$

where

$$f(\omega, \mathbf{k}) = \int \overline{n_{\mathbf{k}}(t) n_{\mathbf{k}}(t + \tau)} e^{-i\omega\tau} d\tau = \int M(\tau, \mathbf{k}) e^{-i\omega\tau} d\tau.$$

From (26) we get

$$M(\tau, \mathbf{k}) = (A / 4\sigma_{\mathbf{k}}\omega_{\mathbf{k}}^2) e^{-\sigma_{\mathbf{k}}|\tau|} \cos \omega_{\mathbf{k}}\tau. \quad (27)$$

In the case of statistical equilibrium we can determine the value of  $A$  from the condition that for  $\tau = 0$  the expression

$$M(\tau, \mathbf{q}) = (2\pi)^{-3} \int M(\tau, \mathbf{k}) e^{-i\mathbf{k}\mathbf{q}} d\mathbf{k}$$

for the correlation function must agree with (19). This condition follows immediately from the definition of the many-time correlation functions.<sup>12,13</sup>

From Eq. (27) it follows that the correlation time is determined by the inverse of the logarithmic decrement  $\sigma_{\mathbf{k}}$ ; this quantity was determined by considering only distant interactions. For small values of  $\mathbf{k}$  one can take the close interactions approximately into account if we introduce the average relaxation time between collisions.

Let us now go over to a consideration of the quantum mechanical case.

In the representation of second quantization the Hamiltonian of a system of particles with central force interactions, which corresponds to the Hamiltonian function (3), has the following form,

$$H = - \int \Psi^+ \frac{\hbar^2}{2m} \Delta_q \Psi dq + \frac{1}{2} \int \Phi(|\mathbf{q} - \mathbf{q}'|) \Psi^+(\mathbf{q}) \Psi(\mathbf{q}) \Psi^+(\mathbf{q}') \Psi(\mathbf{q}') d\mathbf{q} d\mathbf{q}', \quad (28)$$

where  $\Psi^+$  and  $\Psi$  are the quantum mechanical wave functions which satisfy the well known commutation relations. In quantum theory one can define the quantity corresponding to the classical  $N_{\mathbf{q}\mathbf{p}}(t)$  as follows,<sup>5</sup>

$$N_{\mathbf{q}\mathbf{p}}(t) = (2\pi)^{-3} \int \Psi^+(\mathbf{q} - \hbar\theta/2) \Psi(\mathbf{q} + \hbar\theta/2) e^{-i\theta\mathbf{p}} d\theta. \quad (29)$$

We consider first of all the case where we can neglect exchange interactions. By comparison with the expressions for the average values of dynamical quantities which are additive, binary, and so on, we can find relations between the average values of products of the functions  $N_{\mathbf{q}\mathbf{p}}(t)$  and the quantum mechanical distribution functions  $f_1, f_2, \dots$ ,

$$Nf_1(\mathbf{q}, \mathbf{p}, t) = \overline{N_{\mathbf{q}\mathbf{p}}}(t); \quad (30)$$

$$N(N-1)f_2(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t) = \overline{N_{\mathbf{q}\mathbf{p}}N_{\mathbf{q}'\mathbf{p}'}}$$

$$- \int \delta\left(\mathbf{q} + \frac{\hbar\theta}{2} - \left(\mathbf{q}' - \frac{\hbar\theta'}{2}\right)\right) \overline{N\left(\frac{\mathbf{q} - (\hbar\theta/2) + \mathbf{q}' + (\hbar\theta'/2)}{2}, \boldsymbol{\eta}, t\right)} \exp\{i\theta(\boldsymbol{\eta} - \mathbf{p}) + i\theta'(\boldsymbol{\eta} - \mathbf{p}')\} d\theta d\theta' d\boldsymbol{\eta}. \quad (31)$$

These expressions correspond to the classical expressions (6) and (7). If we use the definition of the  $N_{\mathbf{q}\mathbf{p}}(t)$  we can rewrite expression (28) for  $H$  as follows,

$$H = \int \frac{p^2}{2m} N_{\mathbf{q}\mathbf{p}} d\mathbf{q} d\mathbf{p} + \frac{1}{2} \int \Phi(|\mathbf{q} - \mathbf{q}'|) N_{\mathbf{q}\mathbf{p}} N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q} d\mathbf{p} d\mathbf{q}' d\mathbf{p}'. \quad (28')$$

One can obtain an equation for  $N_{\mathbf{q}\mathbf{p}}(t)$  in the quantum mechanical case by using, for instance, the equation for the quantum mechanical wave function  $\Psi$  and the equation has the following form

$$\frac{\partial N_{\mathbf{q}\mathbf{p}}}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial N_{\mathbf{q}\mathbf{p}}}{\partial \mathbf{q}} = \frac{i}{(2\pi)^3} \frac{i}{\hbar} \int \left[ \Phi\left(|\mathbf{q} - \mathbf{q}' - \frac{\hbar\theta}{2}\right| \right) - \Phi\left(|\mathbf{q} - \mathbf{q}' + \frac{\hbar\theta}{2}\right| \right) N_{\mathbf{q}\boldsymbol{\eta}} N_{\mathbf{q}'\mathbf{p}'} e^{i\theta(\boldsymbol{\eta} - \mathbf{p})} d\theta d\boldsymbol{\eta} d\mathbf{q}' d\mathbf{p}'. \quad (32)$$

This equation goes over into the classical equation (5) as  $\hbar \rightarrow 0$ .

Equation (32) has the same form as the quantum mechanical transport equation for the distribution function with a self-consistent field, considered by Silin and the present author.<sup>7,15</sup> As far as the correspondence between these equations is concerned, we can say approximately the same as was said earlier concerning Eq. (5) for the  $N_{\mathbf{q}\mathbf{p}}(t)$  and the classical self-consistent field equation for the distribution function.

The  $N_{\mathbf{q}\mathbf{p}}(t)$  are now, just as in the classical case, random functions of  $\mathbf{q}, \mathbf{p}$ , and  $t$ , and there exists a distribution function  $F(\dots N_{\mathbf{q}\mathbf{p}} \dots)$  for them. Multiplying the average of Eq. (31) by this function one can obtain a set of quantum mechanical equations for  $f_1, f_2$ , and so on.

As in the classical case we shall assume

$$N_{\mathbf{q}\mathbf{p}} = \delta N_{\mathbf{q}\mathbf{p}}(t) + \overline{N_{\mathbf{q}\mathbf{p}}}(t).$$

The equations for  $\delta N_{\mathbf{q}\mathbf{p}}(t)$  and for the mean square deviation have the following form

$$\begin{aligned} \frac{\partial \delta N_{\mathbf{q}\mathbf{p}}}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial \delta N_{\mathbf{q}\mathbf{p}}}{\partial \mathbf{q}} &= \frac{i}{(2\pi)^3} \frac{i}{\hbar} \int \left[ \Phi\left(|\mathbf{q} - \mathbf{q}' - \frac{\hbar\theta}{2}\right| \right) \\ &- \Phi\left(|\mathbf{q} - \mathbf{q}' + \frac{\hbar\theta}{2}\right| \right) \{ \overline{N_{\mathbf{q}\boldsymbol{\eta}} \delta N_{\mathbf{q}'\mathbf{p}'}} + \delta N_{\mathbf{q}\boldsymbol{\eta}} \overline{N_{\mathbf{q}'\mathbf{p}'}} + \delta N_{\mathbf{q}\boldsymbol{\eta}} \delta N_{\mathbf{q}'\mathbf{p}'} \\ &- \overline{\delta N_{\mathbf{q}\boldsymbol{\eta}} \delta N_{\mathbf{q}'\mathbf{p}'}} \} e^{i\theta(\boldsymbol{\eta} - \mathbf{p})} d\theta d\boldsymbol{\eta} d\mathbf{q}' d\mathbf{p}'; \quad \frac{\partial}{\partial t} \overline{\delta N_{\mathbf{q}\mathbf{p}} \delta N_{\mathbf{q}'\mathbf{p}'}} + \frac{\mathbf{p}}{m} \frac{\partial}{\partial \mathbf{q}} \overline{\delta N_{\mathbf{q}\mathbf{p}} \delta N_{\mathbf{q}'\mathbf{p}'}} = \frac{i}{\hbar} \int \left[ \Phi\left(|\mathbf{q} - \mathbf{q}'' - \frac{\hbar\theta}{2}\right| \right) \\ &- \Phi\left(|\mathbf{q} - \mathbf{q}'' + \frac{\hbar\theta}{2}\right| \right) \{ \overline{N_{\mathbf{q}\boldsymbol{\eta}} \delta N_{\mathbf{q}'\mathbf{p}'} \delta N_{\mathbf{q}''\mathbf{p}''}} + \delta N_{\mathbf{q}'\mathbf{p}'} \delta N_{\mathbf{q}\boldsymbol{\eta}} \overline{N_{\mathbf{q}''\mathbf{p}''}} + \overline{\delta N_{\mathbf{q}'\mathbf{p}'} \delta N_{\mathbf{q}\boldsymbol{\eta}} \delta N_{\mathbf{q}''\mathbf{p}''}} \} e^{i\theta(\boldsymbol{\eta} - \mathbf{p})} d\theta d\boldsymbol{\eta} d\mathbf{q}'' d\mathbf{p}''. \end{aligned} \quad (33)$$

In the case when the average values of the  $N_{\mathbf{qp}}(t)$  do not depend on the time and when there is a uniform distribution in coordinate space, that is, when  $\overline{N_{\mathbf{qp}}(t)} = \overline{N}_{\mathbf{p}}$ , the mean square deviation will again [compare Eq. (13)] depend only on  $|t - t'| = \tau$  and on the relative position  $\mathbf{q} - \mathbf{q}' = \mathbf{q}$ .

If we neglect in (33) triple deviations we get the following equation for the function  $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$  defined by (13)

$$\frac{\partial M}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial M}{\partial \mathbf{q}} = \frac{i}{(2\pi)^3 \hbar} \int \left[ \Phi\left(\left|\mathbf{q} - \mathbf{q}' - \frac{\hbar\theta}{2}\right|\right) - \Phi\left(\left|\mathbf{q} - \mathbf{q}' + \frac{\hbar\theta}{2}\right|\right) \right] \overline{N}_{\eta} M(\tau, \mathbf{q}', \mathbf{p}', \mathbf{p}'') e^{i\theta(\eta - \mathbf{p})} d\theta d\eta d\mathbf{q}' d\mathbf{p}''. \quad (34)$$

In the linear approximation we find that Eq. (32) for  $\delta N_{\mathbf{qp}}(t)$  has a solution only provided

$$1 = \nu \left( \frac{|\mathbf{p}|}{\hbar} \right) \int \frac{\overline{N}_{\mathbf{p} + \hbar\mathbf{k}/2} - \overline{N}_{\mathbf{p} - \hbar\mathbf{k}/2}}{\hbar\mathbf{k}\mathbf{p}/m - \hbar\omega} d\mathbf{p}. \quad (35)$$

Expression (35) connects the energy  $\hbar\omega$  with the momentum  $\hbar\mathbf{k}$  of the elementary excitations of the system under consideration and is the counterpart of the classical equation (21) connecting  $\omega$  and  $\mathbf{k}$ . In Ref. 5, we have given for a number of cases the excitation spectrum following from Eq. (35). [In that paper we started from the Hamiltonian function defined by Eq. (2) and not the one defined by Eq. (3) so that the corresponding equation for  $N_{\mathbf{qp}}(t)$  contained an additional term.]

We can obtain an expression for the space-time correlation function either from the solution of Eq. (34) for  $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$ , or by using (35) directly.

If we want to take exchange effects into account, Eq. (30) becomes more complicated, but also in that case can the Hamiltonian (28) be expressed in terms of the  $N_{\mathbf{qp}}$ .

$$H = \int \frac{p^2}{2m} N_{\mathbf{qp}} d\mathbf{q} d\mathbf{p} + \frac{1}{2} \int \left\{ \Phi(|\mathbf{q} - \mathbf{q}'|) \mp (2\pi)^{-3} \nu \left( \frac{|\mathbf{p} - \mathbf{p}'|}{\hbar} \right) \delta(\mathbf{q} - \mathbf{q}') \right\} N_{\mathbf{qp}} N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q} d\mathbf{q}' d\mathbf{p} d\mathbf{p}'. \quad (36)$$

From Eq. (36) it follows that the consideration of exchange effects can be reduced to a change in the interaction energy of the particles in the system.

The equation for  $N_{\mathbf{qp}}(t)$  has in this case the following form,

$$\begin{aligned} \frac{\partial N_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{q}} &= \frac{1}{(2\pi)^3 \hbar} \int \left\{ \Phi\left(\left|\mathbf{q} - \mathbf{q}' - \frac{\hbar\theta}{2}\right|\right) - \Phi\left(\left|\mathbf{q} - \mathbf{q}' + \frac{\hbar\theta}{2}\right|\right) \right\} N_{\mathbf{q}\eta} N_{\mathbf{q}'\mathbf{p}'} e^{i\theta(\eta - \mathbf{p})} d\mathbf{q}' d\mathbf{p}' d\theta d\eta \\ &\mp \frac{1}{(2\pi)^3 \hbar} \int \left\{ \nu\left(\left|\boldsymbol{\eta} - \mathbf{p}' + \frac{\hbar\mathbf{k}}{2}\right|\right) \delta\left(\mathbf{r} - \mathbf{q}' - \frac{\hbar\theta}{2}\right) - \nu\left(\left|\boldsymbol{\eta} - \mathbf{p}' - \frac{\hbar\mathbf{k}}{2}\right|\right) \right. \\ &\left. \times \delta\left(\mathbf{r} - \mathbf{q}' + \frac{\hbar\theta}{2}\right) \right\} N_{\mathbf{r}\eta} N_{\mathbf{q}'\mathbf{p}'} \exp\{i\theta(\boldsymbol{\eta} - \mathbf{p}) + i\mathbf{k}(\mathbf{r} - \mathbf{q})\} d\theta d\eta d\mathbf{k} d\mathbf{r} d\mathbf{q}' d\mathbf{p}'. \end{aligned} \quad (37)$$

For systems of particles, obeying Fermi statistics, we can use a simpler equation, provided the momenta of the particles differ little from the momentum at the top of the Fermi distribution,

$$\begin{aligned} \frac{\partial N_{\mathbf{qp}}}{\partial t} + \left( \frac{\mathbf{p}}{m} - \frac{1}{(2\pi)^3} \frac{\partial}{\partial \mathbf{p}} \int \nu\left(\left|\frac{\mathbf{p} - \mathbf{p}'}{\hbar}\right|\right) \delta(\mathbf{q} - \mathbf{q}') N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}' d\mathbf{p}' \right) \frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{q}} \\ - \frac{\partial}{\partial \mathbf{q}} \int \left\{ \Phi(|\mathbf{q} - \mathbf{q}'|) - \frac{1}{(2\pi)^3} \nu\left(\left|\frac{\mathbf{p} - \mathbf{p}'}{\hbar}\right|\right) \delta(\mathbf{q} - \mathbf{q}') \right\} N_{\mathbf{q}'\mathbf{p}'} d\mathbf{q}' d\mathbf{p}' \frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}} = 0. \end{aligned} \quad (38)$$

Equation (37) has the same form as the quantum mechanical self-consistent field equation for the distribution function which takes the exchange effects into account. From Eqs. (37) and (38) we can by averaging obtain a set of equations for the quantum mechanical distribution functions  $f_1(\mathbf{q}, \mathbf{p}, t)$ ,  $f_2(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t)$  which include exchange. The first equation of this set, which is obtained by averaging Eq. (38) has, for instance, the following form

$$\begin{aligned} \frac{\partial \overline{N_{\mathbf{qp}}}}{\partial t} + \left( \frac{\mathbf{p}}{m} - \frac{1}{(2\pi)^3} \frac{\partial}{\partial \mathbf{p}} \int \nu\left(\left|\frac{\mathbf{p} - \mathbf{p}'}{\hbar}\right|\right) \delta(\mathbf{q} - \mathbf{q}') \overline{N_{\mathbf{q}'\mathbf{p}'}} d\mathbf{q}' d\mathbf{p}' \right) \frac{\partial \overline{N_{\mathbf{qp}}}}{\partial \mathbf{q}} \\ - \frac{\partial}{\partial \mathbf{q}} \int \left\{ \Phi(|\mathbf{q} - \mathbf{q}'|) - \frac{1}{(2\pi)^3} \nu\left(\left|\frac{\mathbf{p} - \mathbf{p}'}{\hbar}\right|\right) \delta(\mathbf{q} - \mathbf{q}') \right\} \overline{N_{\mathbf{q}'\mathbf{p}'}} d\mathbf{q}' d\mathbf{p}' \frac{\partial \overline{N_{\mathbf{qp}}}}{\partial \mathbf{p}} = 0. \end{aligned} \quad (39)$$

An approximate solution of the first two equations of this set in the case of a short range force [ $\Phi(|\mathbf{q} - \mathbf{q}'|) = \nu(0) \delta(\mathbf{q} - \mathbf{q}')$ ] enables us to obtain the transport equation used by Landau in his theory of Fermi-liquids.<sup>17</sup>

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## NUCLEAR REACTIONS IN SUPER-DENSE COLD HYDROGEN

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It is shown that nuclear reactions occurring in cold hydrogen at densities of  $10^4 - 10^6$  g/cc via barrier penetration proceed with a probability which is quite noticeable on an astrophysical scale. This fact puts a limit on the possible compression of cold hydrogen, since a celestial body cannot last more than  $10^8$  years at a density of 0.7 g/cc. Such a density is reached in cold hydrogen under the action of gravitation for a mass close to that of the sun.

It is known<sup>1-3</sup> that the thermonuclear reactions  $p + p = D + e^+ + \tilde{\nu}$ ,  $p + D = \text{He}^3 + \gamma$  occur in stars; in addition, at high temperature we have the reaction  $\text{He}^3 + \text{He}^3 = \text{He}^4 + p + p$ , and at high density the reactions  $\text{He}^3 + e^- = T + \nu$ ,  $T + p = \text{He}^4 + \gamma$ .