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Vacuum Polarization in Mesonic Atoms

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Level shifts in mesonic atoms due to vacuum polarization produced by the nuclear electric field are computed. The recursion relations between the meson wave functions are used to derive equations in closed form for the first six level shifts. The shifts for the first three levels of heavy mesonic atoms are computed with allowance for the finite volume of the nucleus.

1. INTRODUCTION

TO STUDY THE PROPERTIES of mesonic atoms it is important to know the position of the energy levels of the meson. In light mesonic atoms (up to $Z \sim 20$), the energy level is determined principally by nonrelativistic formulas of the Keppler problem of hydrogen-like atoms, in which the electron mass is replaced by a reduced meson mass.[1] The most significant corrections for these formulas are the relativistic corrections, the level shifts due to the vacuum polarization by the electric field of the nucleus,[2] the shift due to the distribution of the positive charge over the volume of the nucleus, and in π-mesoatoms also the shift due to nuclear interaction between the meson and the nucleons of the nucleus.[3] In heavy mesonic atoms the effect of the finite volume of the nucleus on the position of the levels is so considerable that it can no longer be considered a small perturbation. For example, when the volume of the nucleus is taken into account, the energy of the $1S$ muon in μ-meso-lead turns out to have half the value given by the equation for point-like nuclei.[1] In the case of heavy mesonic atoms it therefore becomes necessary to solve from the very beginning for the motion of the meson in the electric field of a nucleus occupying a finite volume. As to the remaining significant corrections to the energy levels of heavy mesonic atoms, to which one must add also the influence of the quadrupole electric moment of the nucleus[4,5] and the polarization (deformation) of the nucleus by the meson,[6,7] these do not exceed $1-2\%$ of the level energy.

While in the hydrogen atom the Lamb shift of the electron energy levels is due fundamentally to the correction for the field electromagnetic mass, and approximately 1/25th of the shift is caused by vacuum polarization, the situation in mesonic atoms is different. The vacuum polarization of electrons and positrons changes the electrostatic potential of the nucleus at a distance on the order of the Compton wavelength of the electron ($\sim 10^{-11}$ cm), regardless of what particle, electron or meson, moves around the nucleus. Since the Bohr radius of the orbit is inversely proportional to the mass of the particle, the radii of the meson orbits are $200-300$ times smaller than the radii of the electron orbits and their dimensions are on the order of $10^{-11}-10^{-12}$ cm. As a result, the meson spends a greater part of its time in a region where the electrostatic potential of the nucleus is changed by the influence of the vacuum polarization, which leads to a considerable level shift. At the same time, the correction for the electromagnetic intrinsic mass, which is inversely proportional to the square of the mass of the moving particle, will be considerably smaller in mesonic atoms than the Lamb shift for electrons, as a consequence of the greater mass of the meson.

The energy level shift due to vacuum polarization is a substantial correction for all mesonic atoms ($\sim 0.1-2\%$ of the energy of the level). The influence of vacuum polarization is noticeable at small Z also for levels with greater orbital momenta l, where the effect of the volume of the nucleus is relatively small. For example, the shift of the $3D_{5/2}$ level of μ-meso-uranium due to vacuum polarization is approximately 10 kev, and that due to the finite volume of the nucleus is only approximately 5 kev. This is clearly seen in Fig. 1 for mesonic atoms with small Z.

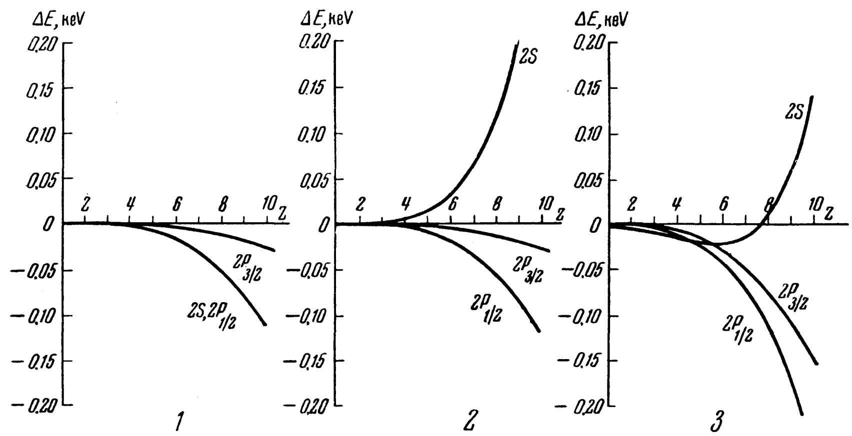

FIG. 1. Dependence of the splitting of the second level in light μ-mesonic atoms on Z, caused by various effects: 1 – relativistic splitting without allowance for the volume of the nucleus and vacuum polarization, 2 – with allowance for the volume of the nucleus, 3 – with additional allowance for vacuum polarization. The zero level represents the nonrelativistic level energy without allowance for the volume of the nucleus.

It is the purpose of this work to calculate the meson level shift due to vacuum polarization in the mesonic atom. For the first six levels, equations were derived for the shift in closed form. The effect of the nuclear volume is accounted for in the calculation of the level shift in heavy mesonic atoms.

2. CALCULATION OF THE LEVEL SHIFTS

To calculate the level shift due to vacuum polarization in mesonic atoms, let us take the Feynman form of the Fourier component of the effective polarization potential induced by an external electric field with a Fourier component of potential $\varphi_0(\mathbf{k})$ (see, for example, Ref. 8).

$$\varphi(\mathbf{k}) = \frac{\alpha}{\pi}\left[\frac{4k_0^2 - 2\mathbf{k}^2}{3\mathbf{k}^2}\left(1 - \frac{\sqrt{4k_0^2 + \mathbf{k}^2}}{|\mathbf{k}|}\sinh^{-1}\frac{|\mathbf{k}|}{2k_0}\right) + \frac{1}{9}\right]\varphi_0(\mathbf{k}), \tag{1}$$

where $k_0 = mc/\hbar$ (m is the electron mass), $\alpha = e^2/\hbar c$ is the fine-structure constant. For a concentrated point nucleus with a Coulomb potential

$$V(r) = eZ/r \tag{2}$$

the Fourier component of the potential will be

$$\varphi_0(\mathbf{k}) = eZ/2\pi^2 k^2. \tag{3}$$

In the coordinate representation we have for the polarization potential

$$\varphi(r) = \frac{4\alpha}{r}\int_0^\infty\left[\frac{4k_0^2 - 2k^2}{3k^2}\left(1 - \frac{\sqrt{4k_0^2 + k^2}}{k}\sinh^{-1}\frac{k}{2k_0}\right) + \frac{1}{9}\right]\varphi_0(k)\, k \sin kr\, dk. \tag{4}$$

The energy level shift will be determined by perturbation theory, averaging the polarization potential over the meson wave functions.

$$\Delta E_{nl} = -e\int |\psi_{nl}(r, \vartheta, \varphi)|^2 \varphi(r)\, d\tau = -e\int_0^\infty R_{nl}^2(r)\,\varphi(r)\, r^2\, dr =$$

$$= -4\alpha e\int_0^\infty I_{nl}(k)\,\varphi_0(k)\left[\frac{4k_0^2 - 2k^2}{3k^2}\left(1 - \frac{\sqrt{4k_0^2 + k^2}}{k}\operatorname{Arsh}\frac{k}{2k_0}\right) + \frac{1}{9}\right]k^2\, dk, \tag{5}$$

$$I_{nl}(k) = \frac{1}{k}\int_0^\infty r R_{nl}^2(r)\sin kr\, dr. \tag{6}$$

Using the notation

$$x = k/2k_0, \quad \varepsilon = nk_0a/Z = nm/\alpha\mu Z, \quad y = kr \tag{7}$$

($a = \hbar^2/\mu e^2$ is the Bohr radius of the meson orbit, μ the reduced meson mass, and m the principal quantum number) expressions (5) and (6) become

$$\Delta E_{nl} = -\frac{4}{3}\alpha e\,(2k_0)^3\int_0^\infty I_{nl}(\varepsilon x)\,\varphi_0(2k_0x)\left[(1 - 2x^2)\,(x - \sqrt{1 + x^2}\sinh^{-1}x) + \frac{x^3}{3}\right]\frac{dx}{x}, \tag{8}$$

$$I_{nl}(\varepsilon x) = \frac{1}{k^3}\int_0^\infty R_{nl}^2\left(\frac{an}{2Z\varepsilon x}y\right) y\sin y\, dy. \tag{9}$$

For a point concentrated nucleus, using the relation $\sinh^{-1}x = \ln(x + \sqrt{1 + x^2})$ and Eq. (3), we can rewrite expression (8) for the level shift as

$$\Delta E_{nl} = -\frac{\alpha^2}{3\pi}mc^2 Z K_{nl}(\varepsilon), \tag{10}$$

$$K_{nl}(\varepsilon) = \frac{4}{\pi}\int_0^\infty I_{nl}(\varepsilon x)\left\{(1 - 2x^2)\,[x - \sqrt{1+x^2}\ln(x + \sqrt{1+x^2})] + \frac{x^3}{3}\right\}\frac{dx}{x^3}. \tag{11}$$

Taking for $R_{nl}(r)$ the expression for the radial portions of the wave functions of the Keppler problem of hydrogen-like atoms (see, for example, Ref. 9), we obtain, using the notation of (7)

$$R_{10}^2 = \frac{1}{2}\left(\frac{k}{\varepsilon x}\right)^3\exp\left\{-\frac{y}{\varepsilon x}\right\},$$
$$R_{20}^2 = \frac{1}{8}\left(\frac{k}{\varepsilon x}\right)^3\left[4 - 4\frac{y}{\varepsilon x} + \left(\frac{y}{\varepsilon x}\right)^2\right]\exp\left\{-\frac{y}{\varepsilon x}\right\},$$
$$R_{21}^2 = \frac{1}{24}\left(\frac{k}{\varepsilon x}\right)^3\left(\frac{y}{\varepsilon x}\right)^2\exp\left\{-\frac{y}{\varepsilon x}\right\},$$
$$R_{30}^2 = \frac{1}{72}\left(\frac{k}{\varepsilon x}\right)^3\left[36 - 72\frac{y}{\varepsilon x} + 48\left(\frac{y}{\varepsilon x}\right)^2 - 12\left(\frac{y}{\varepsilon x}\right)^3 + \left(\frac{y}{\varepsilon x}\right)^4\right]\exp\left\{-\frac{y}{\varepsilon x}\right\},$$
$$R_{31}^2 = \frac{1}{144}\left(\frac{k}{\varepsilon x}\right)^3\left[16 - 8\frac{y}{\varepsilon x} + \left(\frac{y}{\varepsilon x}\right)^2\right]\left(\frac{y}{\varepsilon x}\right)^2\exp\left\{-\frac{y}{\varepsilon x}\right\},$$
$$R_{32}^2 = \frac{1}{720}\left(\frac{k}{\varepsilon x}\right)^3\left(\frac{y}{\varepsilon x}\right)^4\exp\left\{-\frac{y}{\varepsilon x}\right\}. \tag{12}$$

Noting that

$$\exp\{-y/\varepsilon x\} = 2\,(\varepsilon x/k)^3 R_{10}^2, \tag{13}$$

we obtain an expression for the squares of the wave functions of the higher state in terms of the square of the wave function of the 1S state and of its derivatives with respect to the parameter ε:

$$\begin{aligned} R_{20}^2 &= R_{10}^2 + \varepsilon \frac{dR_{10}^2}{d\varepsilon} + \frac{1}{4}\varepsilon^2 \frac{d^2R_{10}^2}{d\varepsilon^2}, \\ R_{21}^2 &= R_{10}^2 + \frac{2}{3}\varepsilon \frac{dR_{10}^2}{d\varepsilon} + \frac{1}{12}\varepsilon^2 \frac{d^2R_{10}^2}{d\varepsilon^2}, \\ R_{30}^2 &= R_{10}^2 + 2\varepsilon \frac{dR_{10}^2}{d\varepsilon} + \frac{4}{3}\varepsilon^2 \frac{d^2R_{10}^2}{d\varepsilon^2} + \frac{1}{3}\varepsilon^3 \frac{d^3R_{10}^2}{d\varepsilon^3} + \frac{1}{36}\varepsilon^4 \frac{d^4R_{10}^2}{d\varepsilon^4}, \\ R_{31}^2 &= R_{10} + \frac{16}{9}\frac{dR_{10}^2}{d\varepsilon}\varepsilon + \frac{19}{18}\varepsilon^2 \frac{d^2R_{10}^2}{d\varepsilon^2} + \frac{2}{9}\varepsilon^3 \frac{d^3R_{10}^2}{d\varepsilon^3} + \frac{1}{72}\varepsilon^4 \frac{d^4R_{10}^2}{d\varepsilon^4}, \\ R_{32}^2 &= R_{10}^2 + \frac{4}{3}\varepsilon \frac{dR_{10}^2}{d\varepsilon} + \frac{1}{2}\varepsilon^2 \frac{d^2R_{10}^2}{d\varepsilon^2} + \frac{1}{15}\varepsilon^3 \frac{d^3R_{10}^2}{d\varepsilon^3} + \frac{1}{360}\varepsilon^4 \frac{d^4R_{10}^2}{d\varepsilon^4}. \end{aligned} \tag{14}$$

If we now introduce into (11) the integrals (9), into which, in turn, we substitute expressions (14), and if we then interchange the order of integration with respect to x and y and the differentiation with respect to ε, we arrive, in the final analysis, to an expression for $K_{nl}(\varepsilon)$ in terms of $K_{10}(\varepsilon)$ and the derivatives of $K_{10}(\varepsilon)$ with respect to ε. Furthermore, the coefficients of the derivatives will coincide with the coefficients of the corresponding derivatives in (14). Thus, having determined the dependence of the shift of the 1S level on ε, we can obtain the shift of the remaining level by differentiating with respect to ε, thus facilitating the computations considerably.

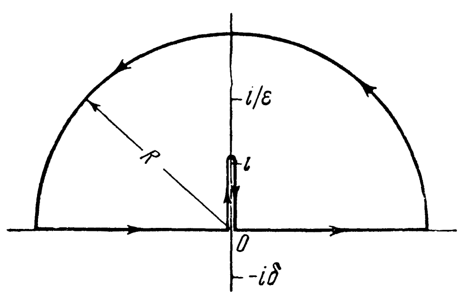

FIG. 2.

According to (9), we have for the 1S state

$$I_{10}(\varepsilon x) = [1 + (\varepsilon x)^2]^{-2}. \tag{15}$$

Then

$$K_{10}(\varepsilon) = \frac{4}{\pi}\int_0^\infty \left\{(1 - 2x^2)\left[x - \sqrt{1 + x^2}\ln\left(x + \sqrt{1 + x^2}\right)\right] + \frac{x^3}{3}\right\}\frac{dx}{x^3(1+\varepsilon^2x^2)^2}. \tag{16}$$

To calculate this integral let us consider the integral

$$K(\delta, \varepsilon) = \frac{4}{\pi\varepsilon^4}\int_C \frac{(1 - 2z^2)\left[z - \sqrt{1 + z^2}\ln\left(z + \sqrt{1 + z^2}\right)\right] + {}^1/_3(z + \delta i)^3}{(z + \delta i)^3(z + i/\varepsilon)^2(z - i/\varepsilon)^2}\,dz, \tag{17}$$

taken in the complex plane along the contour C, shown in Fig. 2, and so chosen that the branch point $z = i$ of the intergrand is located outside the contour. Let us assume that $\varepsilon < 1$, and then the intergrand has a second-order pole inside the contour at the point $z = i/\varepsilon$. Choosing the single-valued branch of the intergrand, we obtain with the aid of the residue theorem

$$\begin{aligned} \lim_{\delta\to 0} K(\delta, \varepsilon) &= 2\frac{4}{\pi\varepsilon^4}\int_0^\infty \frac{(1 - 2x^2)\left[x - \sqrt{1 + x^2}\ln\left(x + \sqrt{1 + x^2}\right)\right] + {}^1/_3 x^3}{x^3(x^2 + \varepsilon^{-2})^2}\,dx + \lim_{\delta\to 0}\frac{4i}{\varepsilon^4}\int_L \frac{(1 - 2z^2)\sqrt{1 + z^2}}{(z + \delta i)^3(z^2 + \varepsilon^{-2})^2}\,dz \\ &= \lim_{\delta\to 0} 2\pi i \operatorname{res}\left\{\frac{4}{\pi\varepsilon^4}\frac{(1 - 2z^2)\left[z - \sqrt{1 + z^2}\ln\left(z + \sqrt{1 + z^2}\right)\right] + {}^1/_3(z + \delta i)^3}{(z + \delta i)^3(z + i/\varepsilon)^2(z - i/\varepsilon)^2}\right\}_{z = i/\varepsilon} \end{aligned} \tag{18}$$

It was taken into account here that the integral along an arc of radius R tends to 0 as $R \to \infty$. The first integral equals $2K_{10}(\varepsilon)$. The integration path L of the second integral consists of the negative part of the real axis from $-\infty$ to 0 and the segment of the imaginary axis from 0 to i. This integral is calculated in a straightforward, although cumbersome, manner. As a result we obtain

$$2K_{10}(\varepsilon) - (3 + 4\varepsilon^2)\pi + \frac{2 - \varepsilon^2 - 4\varepsilon^4}{\varepsilon\sqrt{1-\varepsilon^2}}\pi i = \frac{2}{\varepsilon}\left[-\frac{11}{3} - 4\varepsilon^2 + \frac{2 - \varepsilon^2 - 4\varepsilon^4}{\sqrt{1-\varepsilon^2}}\left(\ln\frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon} + \frac{\pi i}{2}\right)\right]. \quad (19)$$

Hence

$$K_{10}(\varepsilon) = \frac{1}{\varepsilon}\left[-\frac{11}{3} - 4\varepsilon^2 + \left(\frac{3}{2}\varepsilon + 2\varepsilon^3\right)\pi + \frac{2 - \varepsilon^2 - 4\varepsilon^4}{\sqrt{1-\varepsilon^2}}\ln\frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon}\right]. \quad (20)$$

If we now use Eq. (14), in which we substitute K_{nl} and the derivatives K_{10} in place of R^2_{nl} and the derivatives R^2_{10} with respect to ε, we obtain

$$K_{20}(\varepsilon) = \frac{1}{\varepsilon}\left\{-\frac{16}{3} - 14\varepsilon^2 + \pi\left(\frac{3}{2}\varepsilon + 7\varepsilon^3\right) + \frac{3}{4}(1-\varepsilon^2)^{-1} + \frac{9}{4}(1-\varepsilon^2)^{-2}\right.$$
$$\left. + \left[\frac{13}{4} + 4\varepsilon^2 - 14\varepsilon^4 - \frac{9}{4}(1-\varepsilon^2)^{-2}\right]\Phi(\varepsilon)\right\},$$

$$K_{21}(\varepsilon) = \frac{1}{\varepsilon}\left\{-\frac{14}{3} - 10\varepsilon^2 + \pi\left(\frac{3}{2}\varepsilon + 5\varepsilon^3\right) + \frac{5}{4}(1-\varepsilon^2)^{-1} + \frac{3}{4}(1-\varepsilon^2)^{-2}\right.$$
$$\left. + \left[\frac{11}{4} + 2\varepsilon^2 - 10\varepsilon^4 - (1-\varepsilon^2)^{-1} - \frac{3}{4}(1-\varepsilon^2)^{-2}\right]\Phi(\varepsilon)\right\},$$

$$K_{30}(\varepsilon) = \frac{1}{\varepsilon}\left\{-\frac{73}{9} - \frac{92}{3}\varepsilon^2 + \pi\left(\frac{3}{2}\varepsilon + \frac{46}{3}\varepsilon^3\right) + 2(1-\varepsilon^2)^{-1} + \frac{77}{12}(1-\varepsilon^2)^{-2}\right.$$
$$- \frac{65}{6}(1-\varepsilon^2)^{-3} + \frac{35}{4}(1-\varepsilon^2)^{-4} + \left[\frac{16}{3} + \frac{37}{3}\varepsilon^2 - \frac{92}{3}\varepsilon^4 - \frac{5}{12}(1-\varepsilon^2)^{-1}\right.$$
$$\left.\left. - \frac{37}{4}(1-\varepsilon^2)^{-2} + \frac{55}{4}(1-\varepsilon^2)^{-3} - \frac{35}{4}(1-\varepsilon^2)^{-4}\right]\Phi(\varepsilon)\right\},$$

$$K_{31}(\varepsilon) = \frac{1}{\varepsilon}\left\{-\frac{67}{9} - \frac{80}{3}\varepsilon^2 + \pi\left(\frac{3}{2}\varepsilon + \frac{40}{3}\varepsilon^3\right) + \frac{5}{2}(1-\varepsilon^2)^{-1}\right.$$
$$+ \frac{37}{24}(1-\varepsilon^2)^{-2} - \frac{35}{12}(1-\varepsilon^2)^{-3} + \frac{35}{8}(1-\varepsilon^2)^{-4} + \left[\frac{29}{6} + \frac{31}{3}\varepsilon^2 - \frac{80}{3}\varepsilon^4\right.$$
$$\left.\left. - \frac{49}{24}(1-\varepsilon^2)^{-1} - \frac{17}{8}(1-\varepsilon^2)^{-2} + \frac{35}{8}(1-\varepsilon^2)^{-3} - \frac{35}{8}(1-\varepsilon^2)^{-4}\right]\Phi(\varepsilon)\right\},$$

$$K_{32}(\varepsilon) = \frac{1}{\varepsilon}\left\{-\frac{55}{9} - \frac{56}{3}\varepsilon^2 + \pi\left(\frac{3}{2}\varepsilon + \frac{28}{3}\varepsilon^3\right) + \frac{19}{10}(1-\varepsilon^2)^{-1}\right.$$
$$+ \frac{101}{120}(1-\varepsilon)^{-2} + \frac{5}{12}(1-\varepsilon^2)^{-3} + \frac{7}{8}(1-\varepsilon^2)^{-4} + \left[\frac{23}{6} + \frac{19}{3}\varepsilon^2\right.$$
$$\left.\left. - \frac{56}{3}\varepsilon^4 - \frac{37}{24}(1-\varepsilon^2)^{-1} - \frac{5}{8}(1-\varepsilon^2)^{-2} - \frac{1}{8}(1-\varepsilon^2)^{-3} - \frac{7}{8}(1-\varepsilon^2)^{-4}\right]\Phi(\varepsilon)\right\}, \quad (21)$$

where

$$\Phi(\varepsilon)\frac{1}{\sqrt{1-\varepsilon^2}}\ln\frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon} \qquad (\varepsilon < 1). \quad (22)$$

These equations can be continued analytically to include the case when $\varepsilon > 1$. In this case (22) it becomes

$$\Phi(\varepsilon) = \frac{1}{\sqrt{\varepsilon^2 - 1}} \tan^{-1} \sqrt{\varepsilon^2 - 1}. \tag{23}$$

We finally determine the vacuum polarization shift of the first six energy levels of the meson in mesonic atoms with Eqs. (10) and (20)–(23). One need merely take into account that according to (7) ε has different values for states with different principal quantum numbers n. Figs. 3 and 4 show the Z-dependence of the level shifts for π and μ mesonic atoms, calculated with the aid of these equations.

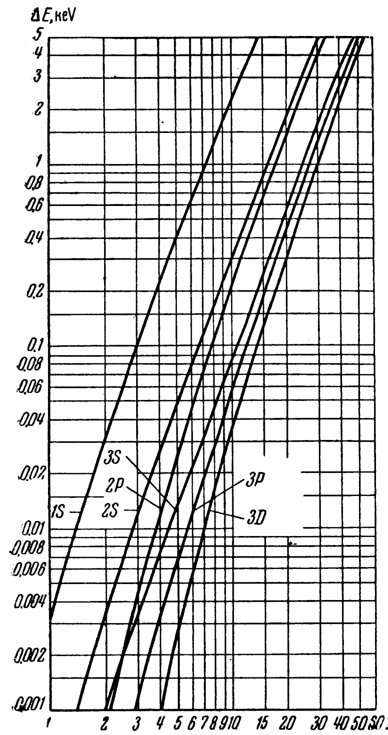

FIG. 3. Dependence of the value of the level shift of π-mesonic atoms on Z, caused by vacuum polarization. The mass of the pion is 272.5 m_e.

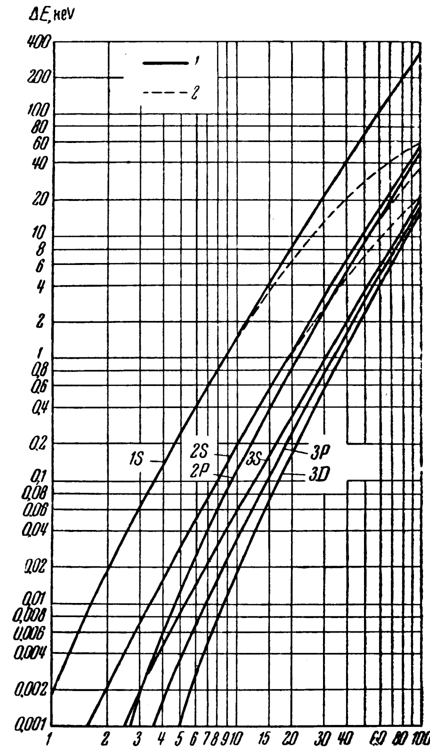

FIG. 4. Dependence of the value of the level shift of μ-mesonic atoms on Z, resulting from vacuum polarization: 1 — without allowance for the volume of the nucleus, 2 — with allowance for the volume of the nucleus.

3. ALLOWANCE FOR THE VOLUME OF THE NUCLEUS

Taking account of the volume distribution of the charge of the nucleus reduces considerably the effect of vacuum polarization in high-Z mesonic atoms (see Refs. 10 and 11). The fact that the nucleus is not concentrated in a point leads, on one hand, to a change in the potential of the electrostatic field of the nucleus, and on the other hand to a change in the meson wave functions. If we take the cause of vacuum polarization to be a nucleus with a charge uniformly distributed within a sphere of radius R_0, there will be an oscillator potential inside the nucleus and a Coulomb potential on the outside

$$\begin{aligned} V(r) &= \frac{eZ}{R_0}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R_0}\right)^2\right] \quad (r < R_0), \\ V(r) &= eZ/r \quad (r > R_0). \end{aligned} \tag{24}$$

The Fourier component of such a potential will be

$$\varphi_0(k) = \frac{eZ}{2\pi^2 k^2} \frac{3}{R_0^2 k^2}\left(\frac{\sin kR_0}{kR_0} - \cos kR_0\right). \tag{25}$$

To account for the change in the meson wave functions, caused by the finite nuclear dimensions, let us turn to the variational method. Let us take for the states 1S, 2S, and 2P respectively orthonormalized trial wave functions

$$R_{10}(r) = 2\left(q\frac{Z}{a}\right)^{3/2}\exp\left\{-q\frac{z}{a}r\right\},$$

$$R_{20}(r) = \frac{2\sqrt{3}s\left(s\frac{Z}{a}\right)^{3/2}}{\sqrt{s^2-sq+q^2}}\left[1-\frac{1}{3}\frac{Z}{a}(s+q)r\right]\exp\left\{-s\frac{Z}{a}r\right\}, \tag{26}$$

$$R_{21}(r) = \frac{2}{\sqrt{3}}\left(t\frac{Z}{a}\right)^{5/2}r\exp\left\{-t\frac{Z}{a}r\right\}$$

with variational parameters q, s, and t. With the aid of the radial portion of the Schroedinger equation with potential (24)

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\frac{2\mu}{\hbar^2}E+\frac{3Z}{aR_0}-\frac{Z}{aR_0^3}r^2-\frac{l(l+1)}{r^2}\right)R=0 \quad (r<R_0),$$

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\frac{2\mu}{\hbar}E+\frac{2Z}{ar}-\frac{l(l+1)}{r^2}\right)R=0 \quad (r>R_0), \tag{27}$$

we obtain the following expression for the energy

$$E=-\frac{\hbar}{2\mu}\left\{-\int_0^\infty\left[\left(\frac{dR}{dr}\right)^2r^2+l(l+1)R^2\right]dr+\int_0^{R_0}\left(\frac{3Z}{aR_0}-\frac{Z}{aR_0^3}r^2\right)R^2r^2dr+\int_{R_0}^\infty\frac{2Z}{a}R^2rdr\right\}. \tag{28}$$

Substituting the trial functions (26) into (28) leads to expressions for the energy in terms of the variational parameters

$$E_{10}=-\frac{\hbar^2}{2\mu}\left(\frac{Z}{a}\right)^2q\left[-q+\frac{3}{pq}-\frac{3}{p^3q^3}+\left(\frac{3}{pq}+\frac{6}{p^2q^2}+\frac{3}{p^3q^3}\right)\exp\{-2pq\}\right],$$

$$E_{20}=-\frac{\hbar^2}{2\mu}\left(\frac{Z}{a}\right)^2\left\{\frac{3}{p}-\frac{s^2}{3}+\frac{s}{s^2-sq+q^2}\left[-2s^3-\frac{3s^2+15q^2}{2p^3s^3}\right.\right. \tag{29}$$

$$\left.\left.+\left(ps^3+2ps^2q+psq^2-s^2+4sq+5q^2+\frac{12q^2+3sq}{ps}+\frac{15q^2+3s^2}{p^2s^2}+\frac{15q^2+3s^2}{p^3s^3}\right)\exp\{-2qs\}\right]\right\},$$

$$E_{21}=-\frac{\hbar^2}{2\mu}\left(\frac{Z}{a}\right)^2t\left[-t+\frac{3}{pt}-\frac{15}{2p^3t^3}+\left(pt+5+\frac{12}{pt}+\frac{15}{p^2t^2}+\frac{15}{2p^3t^3}\right)\exp\{-2pt\}\right].$$

Here $p = R_0Z/a$. The values of the parameters q, s and t are found, as usual, from the minimum-energy condition. The transcendental equations thus obtained can be solved numerically. The table gives values for the variational parameters and for the energy levels of several μ-mesonic atoms (it was assumed in the calculation that the mass of a μ-meson is 270 m, and the nuclear radius $R_0 = 1.2 \times 10^{-13} A^{1/3}$ cm).

Using the functions (26) we obtain by means of formula (9) expressions for I_{nl} with allowance for the volume of the nucleus

$$I_{10}=1/[1+(\varepsilon_qx)^2]^2,$$

$$I_{20}=\frac{3s^2}{s^2-sq+q^2}\left\{\frac{1}{[1+(\varepsilon_sx)^2]^2}-\frac{s+q}{3s}\frac{3-(\varepsilon_sx)^2}{[1+(\varepsilon_sx)^2]^3}+\frac{(s+q)^2}{3s^2}\frac{1-(\varepsilon_sx)^2}{[1+(\varepsilon_sx)^2]^4}\right\}, \tag{30}$$

$$I_{21}=[1-(\varepsilon_tx)^2]/[1+(\varepsilon_tx)^2]^4,$$

where

$$\varepsilon_q = k_0 a / qZ, \quad \varepsilon_s = k_0 a / sZ, \quad \varepsilon_t = k_0 a / tZ.$$

Element	Variational parameters			Energy, Mev				
				Point nucleus		Variational method		
	q	s	t	E_{10}	E_{20}, E_{21}	E_{10}	E_{20}	E_{21}
Si_{14}^{28}	0.9642	0.4918	0.5000	0.550	0.137	0.536	0.136	0.137
Zn_{30}^{65}	0.8376	0.4558	0.4994	2.530	0.632	2.202	0.590	0.632
Sb_{51}^{121}	0.6685	0.3938	0.4926	7.317	1.829	5.183	1.525	1.817
Pb_{82}^{208}	0.5003	0.3197	0.5480	18.923	4.731	9.986	3.316	4.511

The expressions for the level shifts are then obtained by inserting (25) and (30) into (18). The values of the level shifts, obtained therefrom by numerical integration with allowance for the volume of the nucleus, are shown dotted in Fig. 4. By way of an example let us indicate that for μ-meso-lead the shift in the 1S level, due to polarization of vacuum, is 217 kev without allowance for the volume of the nucleus, and nearly 53 kev with allowance for the volume of the nucleus; for the 2S and 2P levels the shifts are 37 and 33 kev respectively without allowance for the volume of the nucleus, and 17 and 28 kev with allowance for the nucleus.

To check how good the approximate solution obtained by the variational method is, we obtained for Eq. (27), by numerical integration by the Runge-Kutta method, the eigenvalue of the energy and the wave function of the 1S state of μ-meso-lead, for the same values of the meson mass and nuclear radius. The eigenvalue of the energy of the 1S level was found to be 10.412 Mev, *i.e.*, approximately 4% greater than the value obtained by the variational method. Fig. 5 shows that the variational wave function is a better approximation than the wave function of the Keppler problem.

It must be noted that in all the above calculations of the mesonic atom level shifts due to vacuum polarization, we took into consideration only terms of the order αZ. As shown in Ref. 12, accounting for terms of higher orders gives for a point nucleus, even in the case of μ-meso-uranium, a shift which amounts to less than 0.02% of the energy level. It is evident that allowance for the volume of the nucleus can only reduce this figure.

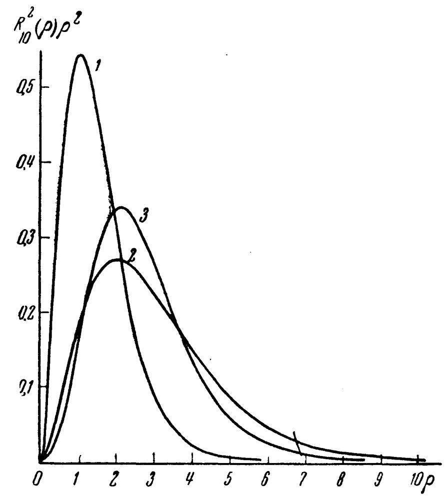

FIG. 5. Radial probability density of finding a meson in μ-meso-lead in state 1S, determined by various methods: 1 — without accounting for the volume of the nucleus [wave function of the Keppler problem $R(\rho) = 2e^{-\rho}$], 2 — with allowance for the volume of the nucleus [variational wave function $R(\rho) = 2q^{3/2}e^{-q\rho}$, $q = 0.5003$], 3 — with allowance for the volume of the nucleus (numerical solution); $\rho = rZ/a$.

The influence of vacuum polarization on the position of the energy levels in mesonic atoms was actually discovered experimentally. In many works on the determination of the masses of π- and μ-mesons using spectra of mesonic atoms it became necessary, to obtain mass values that are in agreement with data of other experiments to compare theory with experiment, to take into account the vacuum polarization level shift in the mesonic atoms.[13–14] Vacuum polarization is also accounted for in investigations of the interaction between the π-meson in the mesonic atom with the nucleons of the nucleus.[15]

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On the Structure of the Front of Strong Shock Waves in Gases

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The internal structure of the front of strong shock waves is investigated, taking account of radiation. Approximate solutions of the equations of the mode are found. Profiles of the hydrodynamic quantities, density and radiation flux, are constructed.

ONE OF THE METHODS of study of shock waves in gases (in particular, in air) is photometric measurement of the brightness of the wave front. In a certain amplitude interval, the shock wave front radiates like a black body. Consequently, it is possible to determine the temperature behind the wave front directly, by photometry. Combined with the measurement of another parameter of the wave, for example, its velocity, this allows us to make some suppositions concerning the thermodynamic functions of the gas being studied. The question arises, up to what amplitudes does the visible temperature coincide with the temperature behind the shock wave, and what is its dependence on the actual temperature behind the front when the latter reaches tens and hundreds of thousands of degrees, since at the present time such powerful shock waves are becoming the subject of experimental investigation.[1] This question leads, first of all, to the problem of the internal structure of a shock wave front, taking account of radiation.

This problem was investigated by Prokof'ev,[2] who obtained correct integrals of the approximate equations in the separate regions in which the variables are continuous. However, as a result of an erroneous analysis of the equations, he joined these solutions in an incorrect way, which led to the continuity of the hydrodynamic variables in the wave. Prokof'ev's error was pointed out by Zel'dovich,[3] who gave a correct qualitative analysis of the approximate equations of the mode, and proved that there is a discontinuity of the hydrodynamic variables in the shock wave.

In the present article, approximate solutions are found of the equations of the mode, encompassing a broad interval of shock wave amplitudes, as well as