Concerning some General Relations of Quantum Electrodynamics

E. S. FRADKIN
P. N. Lebedev Institute of Physics, Academy of Sciences, USSR
(Submitted to JETP editor March 9, 1955)
J. Exper. Theoret. Phys. USSR 29, 258-261
(August, 1955)

IN this letter, the derivations of several general relationships connected with the qauge invariance of quantum electrodynamics are presented.

1. In quantum electrodynamics, the following relation holds in the presence of the photon field G:

$$G^{-1}(p, p'+s) - G^{-1}(p-s, p') = -s_\mu \frac{\delta G^{-1}(pp')}{\delta e A_\mu(s)} = s_\mu \Gamma_\mu (pp's). \quad (1)$$

In fact, according to reference 1, Eqs. (7)-(12), we have,

$$G^{-1}(p, p'+k) - G^{-1}(p-k, p') = \hat{k}\delta(p-p'-k) - \frac{e^2}{(2\pi)^4 i}\int \gamma_\mu \{G(p+s, s_1)\Gamma_\nu(s_1, p'+k, s_2) - G(p-k+s, s_1)\Gamma_\nu(s_1 p' s_2)\} \times D_{\nu\mu}(s_2 s)\, d^4s\, d^4s_1\, d^4s_2. \quad (2)$$

In addition, it can be stated that:

$$k_\mu \Gamma_\mu(pp'k) = \hat{k}\delta(p-p'-k) + \frac{e^2}{(2\pi)^4 i}\int \gamma_\rho \Bigg[\Bigg\{G(p+s, s_3)\, k_\mu \Gamma_\mu(s_1 s_4 k)\, G(s_2 s_1) \times \Gamma_\nu(s_1 p' s_2)\, d^4s_3\, d^4s_4 + G(p+s, s_1) \times k_\mu\left(-\frac{\delta^2 G^{-1}(s_1 p')}{\delta e A_\mu(k)\,\delta e A_\nu(s)}\right)\Bigg\} D_{\nu\rho}(s_2 s) + G(p+s, s_1) \times \Gamma_\nu(s_1 p' s_2)\, k_\mu \frac{\delta D_{\nu\rho}(s_2 s)}{\delta e A_\mu(k)}\Bigg] d^4s_1\, d\, d^4s_2\, d^4s. \quad (3)$$

Making use of Eq. (1) and its functional derivative with respect to A, and taking into account the fact that in Eq. (3) the last term is equal to zero, because of the transverse nature of the polarization operator [see Eq. (5) below], we have

$$k_\mu \Gamma_\mu(pp'k) = \hat{k}\delta(p-p'-k) - \frac{e^2}{(2\pi)^4 i}\int \gamma_\nu \{G(p+s, s_1)\Gamma_\nu(s, p'+k, s_2) - G(p-k+s, s_1)\Gamma_\nu(sp's_2)\} D_{\nu\mu}(s_2 s)\, d^4s\, d^4s_1\, d^4s_2. \quad (4)$$

It can be seen from Eqs. (2) and (4) that Eq. (1) can be regarded as one of the consequences of the exact system of equations for Green's function. Equation (1) permits us to conclude that the polarization operator is transverse in nature, and consequently the rest mass of the photon is zero. Indeed, referring to the expression for the polarization operator given in reference 1 [Eq. (11)], and making use of Eq. (1), we obtain

$$k_\nu P_{\mu\nu}(p, k) = \frac{e^2}{(2\pi)^4 i}\,\mathrm{Sp}\left\{\int \gamma_\mu G(p+s, s_1) \times [G^{-1}(s_1, s_2 - k) - G^{-1}(s_1+k, s_2)]\, G(s_2 s)\, d^4s_2\, d^4s_1\, d^4s\right\} = 0. \quad (5)$$

Relation (1) is equivalent to the infinite series expansion of the relation for $J = 0$. The terms of the expansion can be obtained from Eq. (1) by means of successive differentiation with respect to $A_{\nu_n}(S_n)$, or (see the work of Green[2])

$$G^{-1}(p) - G^{-1}(p-s) = s_\mu \Gamma_\mu(p, p-s, s); \quad (6)$$

$$\Gamma^{(n)}_{\nu_1\ldots\nu_n}\left(p-k, p-k-\sum_{m=1}^{n} s_m, s_1\ldots s_n\right) - \Gamma^{(n)}_{\nu_1\ldots\nu_n}\left(p, p-\sum_{m=1}^{n} s_m, s_1\ldots s_n\right) = k_\mu \Gamma^{(n+1)}_{\mu\nu_1\ldots\nu_n}\left(p, p-k-\sum_{m=1}^{n} s_m, k, s_1\ldots s_n\right),$$

$$\partial G^{-1}(p)/\partial p_\mu = \Gamma_\mu(p,p,0),$$

$$-\frac{\partial^2 G^{-1}(p)}{\partial p_\mu \partial p_\nu} = \left.\frac{\partial \Gamma_\mu(p, p-s, s)}{\partial s_\nu}\right|_{s=0} + \left.\frac{\partial \Gamma_\nu(p, p-s, s)}{\partial s_\mu}\right|_{s=0},$$

where

$$\Gamma^{(n)}_{\nu_1\ldots\nu_n}\left(p, p-\sum_{m=1}^{n} s_m, s_1\ldots s_n\right) = \left(\prod_{m=1}^{n} \frac{\delta}{\delta e A_{\nu_m}(s_m)}\right)\left(-G^{-1}(p,p')\right)\Big|_{J=0}$$

2. Relations (1) and (6) simplify to a considerable degree the calculations concerned with the longitudinal component of the electromagnetic field. Indeed, Green's function of the photons can be represented as follows:

$$D_{\mu\nu}(k) = \frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{k^4} d(k^2) + \frac{k_\mu k_\nu}{k^4} d_l(k^2). \quad (7)$$

Substituting Eq. (7) into the mass operator of the electron, we obtain the following expression for Green's function of the electron for $J = 0$:

$$\{\hat{p} - m - \sum_t (p)\} G(p) = 1 + \sum_l (p) G(p); \quad (8)$$

$$\sum_t (p) = \frac{e^2}{(2\pi)^4 i} \int \gamma_\mu G(p+s) \Gamma_\nu (p+s, p, s) \times \frac{s^2 \delta_{\mu\nu} - s_\mu s_\nu}{s^4} d(s^2) d^4 s,$$

$$\sum_l (p) \quad (9)$$

$$= \frac{e^2}{(2\pi)^4 i} \int \gamma_\mu G(p+s) \Gamma_\nu (p+s, p_1, s) \frac{s_\mu s_\nu}{s^4} d_l(s^2) d^4 s.$$

Taking into account Eq. (6), we have,

$$\sum_l (p) G(p) = -\frac{e^2}{(2\pi)^4 i} \int \gamma_\mu G(p+s) \frac{s_\mu}{s^4} d^4 s. \quad (10)$$

It is known[3] that if the diagram with intersecting photon lines is not taken into consideration, the term Σ_t will not contain any infinities, and consequently, the limiting value of the approximation obtained in this manner will be determined by the longitudinal field component. In this case, it is not necessary to consider the non-linear equations given by Landau[3]; rather, it is sufficient to solve the linear equation

$$(\hat{p} - m) G(p) = 1 - \frac{e^2}{(2\pi)^4 i} \int G(p+s) \frac{\hat{s}}{s^4} d_l(s) d^4 s. \quad (11)$$

Following Landau, let us find the limiting value of $G(p)$ for large values of p^2 in the form $G(p) = \beta(p)/\hat{p}$; then, (11) can be written as:

$$\beta(r) = 1 - \frac{e^2}{16\pi^2} \int_\xi^\eta \beta(z) d_l(z) dz, \quad (12)$$

where $\xi = \ln(-p^2/m^2)$, $\eta = \ln(-L^2/m^2)$, L is the upper cut-off momentum, which must be allowed to go to infinity in the limiting case. Eq. (12) yields the solution which was obtained in reference 3,

$$\beta(\xi) = \exp\left\{-\frac{e^2}{16\pi^2} \int_\xi^\eta d_l(z) dz\right\}. \quad (13)$$

Furthermore, it follows from Eq. (6) that if one of the momenta of the Γ_μ-function is considerably larger than the other, and larger than m^2, the limiting value of Γ_μ is given by:

$$\Gamma_\mu (p, s) \approx \gamma_\mu \beta^{-1}(\xi), \quad (14)$$

with logarithmic accuracy*; here, ξ represents the larger momentum.

Keeping in mind that the polarization operator is independent of the longitudinal component of the electromagnetic field (proof presented below), it is easy to see that in order to obtain the limiting value of Green's function of the photons in our approximation, it is sufficient to calculate the polarization operator by methods of the perturbation theory and substitute it into the equation of Green's function of the photons. Simple calculations yield:

$$d(k) = \left[1 + \frac{e^2}{12\pi^2} \ln \frac{L^2}{k^2}\right]^{-1}. \quad (15)$$

In this manner we can find all the limiting values found in reference 3.

3. Making use of references 4 and 5, it is easy to obtain an explicit expression for the changes in the S-matrix caused by the introduction of an interaction of the type $j_\mu \partial\phi/\partial x_\mu$, where j_μ is the electronic current. In particular, letting

$$A_\mu = A_\mu^t + A_\mu^l = A_\mu^t + \frac{\partial \varphi}{\partial x_\mu}$$

(ϕ is some real function), we can transform exactly the longitudinal component of the interaction.

The expectation value of the S-matrix for the vacuum-vacuum states acquires then the following form, in notation of reference 5:

$$\frac{\langle S \rangle_0}{c} = \exp\left\{ie \int\int_0^1 d\lambda \,\mathrm{Sp}\, \gamma_\mu G\left(xx\lambda \frac{\delta}{\delta j^t}\right) \frac{\delta}{\delta j^t} d^4 x\right.$$

$$+ \sum_{k=0}^{\infty} \frac{i^n}{k!} \int dx_1 \, dx_1' \, dx_2 \, dx_2' \ldots dx_k' dx_k \bar{\eta}(x_2)$$

$$\times G\left(x_2 x_1' \frac{\delta}{\delta j^t}\right) \eta(x_1' \ldots) + \bar{\eta}(x_k) G\left(x_k x_k' \frac{\delta}{\delta j^t}\right) \eta(x_k')$$

$$\times \exp\left(\frac{ie}{2}\right\{ \sum_{(n,m)\, m+n=1}^{k} (e) [\Delta_l (x_m - x_n)$$

$$+ \Delta_l (x_m' - x_n') - 2\Delta_l (x_m - x_n')$$

$$- 2 \sum_{m=1}^{k} \int \left[\frac{\partial \Delta_l (x_m - z)}{\partial z_\mu} - \frac{\partial \Delta_l (x_m' - z)}{\partial z_\mu}\right]\Big) j_\mu^l (z) d^4 z \Big\}$$

$$\times \exp\left\{i \int\left[j_\mu^t (x) \frac{D_{\mu\nu}^t (x-y)}{2} j_\nu^t (y) \right.\right. \quad (16)$$

$$\left.\left. + j_\mu^l (x) D_{\mu\nu}^l (x-y) j_\mu^l (y) \right] d^4 x d^4 y\right\},$$

where

$$\Delta_l(xx') = -i\langle\varphi(x)\varphi(x')\rangle_t,$$

$$D^t_{\mu\nu}(x-y) = \frac{1}{(2\pi)^4}\int \frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{k^4} e^{ik(x-y)}d^4k; \quad D^l_{\mu\nu}$$

$$= \frac{1}{(2\pi)^4}\int \frac{k_\mu k_\nu}{k^2}\Delta_l(k^2)e^{ik(x-y)}d^4k,$$

$\Delta_l(k^2)$ is any positive function of k^2 and $j_\mu = j^t_\mu + j^l_\mu$ are the external sources of the transverse and the longitudinal components of the interaction with the photons (the interaction with an external field being given by $j_\mu A_\mu = j^t_\mu A^t_\mu + j^l_\mu A^l_\mu$).

The use of functional differentiation with respect to the sources of the photon or the electron fields reveals an obvious dependence of Green's functions on the longitudinal component of the field. In particular, from Eq. (16) we obtain the following relations:

$$G(x,y) = G(x,y)_0 \exp\{-ie^2(\Delta_l(0) - \Delta_l(x-y)\},$$

$$\frac{\delta G(x,y)}{\delta j_\mu(z)} = \frac{\delta G(x,y)}{\delta j^t_\mu(z)} + \frac{\delta G(x,y)}{\delta j^l_\mu(z)} = \left(\frac{\delta G(x,y)}{\delta j^t_\mu(z)}\right)_0$$

$$\times \exp\{-ie^2[\Delta_l(0) - \Delta_l(x-y)]\} + ieG(x,y)_0 \times \left[\frac{\partial\Delta_l(x-z)}{\partial z_\mu} + \frac{\partial\Delta_l(y-z)}{\partial z_\mu}\right]$$

where the index 0 denotes that the given quantity represents the unperturbed solution, where the longitudinal component of the interaction is totally absent.

Since the quantity $\delta G(x,x)/\delta j_\mu(z)$ is not dependent on the longitudinal component of the field, according to Eq. (18), the polarization operator is also independent of the longitudinal component. It follows from Eq. (17) that no gauge transformation is capable of removing the infinities which are caused by the transverse component of the interaction, and, therefore, in general, the most convenient choice of a gauge transformation is $\Delta_l = 0$ (see also reference 3).

* *Translator's note:* The expression "logarithmic accuracy" denotes the following:

$$P/\xi \geq \log \xi, \; P \ll \xi.$$

[1] E. S. Fradkin, J. Exper. Theoret. Phys. USSR 26, 752 (1954)

[2] H. S. Green, Phys. Rev. 95, 548 (1954)

[3] L. D. Landau et al, Dokl. Akad. Nauk SSSR 95, 773 (1954)

[4] E. S. Fradkin, Dokl. Akad. Nauk SSSR 98, 47 (1954)

[5] E. S. Fradkin, Dokl. Akad. Nauk SSSR 100, 897 (1955)

Translated by G. Makhov
188

Note on Subsequent Transitions in Meson-Atoms

M. I. PODGORETSKII
P. N. Lebedev Institute of Physics, Academy of Sciences, USSR

(Submitted to JETP editor January 20, 1955)
J. Exper. Theoret. Phys. USSR 29, 374-375 (September, 1955)

MUCH experimental work has been done on the study of γ-quanta which are emitted during various transitions in meson atoms, and great precision has been achieved in measuring the energies of these γ-quanta. Of the various transitions which have been studied some are subsequent, for instance $3d \to 2p$ and $2p \to 1s$. However, in practice, the possible connection between the transitions has not been experimentally investigated. During the subsequent radiative transitions in meson atoms noticeable angular correlation between the directions of emission of the γ-quanta should be observed in many cases. According to the general laws for subsequent transitions (see reference 1, Ch. VII) the angular correlation is determined only by the knowledge of the total momenta of the initial, final and intermediate states. Since the orbital momenta of the levels of the meson atoms are well-known, in the given case the correlation is determined only by the magnitude of the meson spin. Thus there exists a possibility of the direct determination of the meson spin by measuring the angular correlation. In principle, this applies to the negative mesons of arbitrary type, and in particular, to the heavy mesons.

As an illustration let us consider μ mesons. By